

SAMPLING AND INTERPOLATION ON SOME TWO-STEP NILPOTENT LIE GROUPS

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ABSTRACT. We prove the existence of sampling space with the interpolation property on a fairly large class of step two nilpotent Lie groups. Let N be a simply connected, connected, two step nilpotent Lie group with Lie algebra \mathfrak{n} such that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{z}$, the algebras $\mathfrak{a}, \mathfrak{b}$ are commutative and have the same dimension. Moreover we assume that $\det([X_i, Y_j])_{1 \leq i, j \leq d}$ is a non-vanishing polynomial on the commutator ideal $[\mathfrak{a}, \mathfrak{b}]$. We prove the existence of a set $\Gamma \subset N$, and a left-invariant subspace \mathbf{H} in $L^2(N)$ such that \mathbf{H} is a sampling space with respect to Γ and \mathbf{H} has the interpolation property. A sinc-type function is constructed explicitly. Also, many examples are computed to support our results.

1. INTRODUCTION

A function $f \in L^2(\mathbb{R})$ is bandlimited to $[-a, a]$ if the support of its Fourier transform is contained in $[-a, a]$. When $a = 1/2$, we call such space the Paley-Wiener space in $L^2(\mathbb{R})$. It is a well-known fact that bandlimited spaces are invariant under the action of the left regular representation of \mathbb{R} . Another fact which will be of special interest in this paper is the following. The integer translates of the Fourier inverse transform of the characteristic function of the interval $[-1/2, 1/2]$ forms an orthonormal basis for the Paley-Wiener space. As a result, we have a precise algorithm for reconstructing an arbitrary bandlimited function f from countably many sample values $\{f(k) : k \in \mathbb{Z}\}$. Let

$$\mathbf{H} = \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \text{supp } (\hat{f}) \subseteq [-1/2, 1/2] \right\}.$$

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For all $f \in \mathbf{H}$,

$$(1) \quad f(x) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

and the series converges unconditionally in L^2 -norm. Moreover

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} |f(k)|^2.$$

The function $\frac{\sin(\pi t)}{\pi t}$ is called the **sinc** function. The series in (1) gave rise to the uniform sampling theory of Shannon which plays an important role in digital signal processing.

The basic mathematical question: which locally compact groups G and which closed left-invariant subspaces of $L^2(G)$ admit sinc-type functions? is an open question. In fact, if \mathbb{R} is replaced with a nonabelian locally compact group G then it is not clear in general what should be the analogue of \mathbb{Z} , and how to generalize the concept of bandlimitation. Pesenson [9] introduced a geometric definition of bandlimitation on stratified Lie groups. He defines a bandlimited function to be a function whose sub-Laplacian transform is supported on a bounded subset of the real line. Führ and Gröchenig [7] also use Pesenson's definition for stratified nilpotent in [7]. Additionally, they replace \mathbb{Z} with some quasi-lattice in G . Their results are very general and very precise. However, they were only able to construct frames as opposed to orthonormal bases. In [4], Dooley investigated the same question for motion groups of the type $\mathbb{R}^k \rtimes K$ for a compact group K . Currey and Mayeli specialized to the 3-dimensional Heisenberg group in [1], and they define a function to be bandlimited whenever its group Fourier transform is supported on a bounded subset of the spectrum of the left regular representation of the Heisenberg group. We remark that this definition is much closer and natural to the classical definition of bandlimitation than the definition used in [7]. Also, for a class of two-step nilpotent Lie groups, we have obtained some specific conditions in [8] for the construction of Parseval frames in bandlimited spaces using a similar definition to the one used in [1]. We would like to remark that the work by Currey and Azita in [1] suggest that similar results could also be obtained for other non commutative type of nilpotent Lie groups.

In this paper, we consider the same class of nilpotent Lie groups studied in [8]. It is a class of two-step nilpotent Lie groups which is a natural generalization of the Heisenberg groups. Here is a precise

description of this class of groups. Let N be a non-commutative connected, simply connected nilpotent Lie group with Lie algebra \mathfrak{n} over the reals. Fix a Jordan Hölder basis for the Lie algebra of N such that $\mathfrak{a} = \mathbb{R}\text{-span} \{X_1, \dots, X_d\}$, and $\mathfrak{b} = \mathbb{R}\text{-span} \{Y_1, \dots, Y_d\}$. Moreover, we assume that

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}, [\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{z},$$

the algebras \mathfrak{a} and \mathfrak{b} are commutative and have the same dimension. Finally, we assume that the determinant of the matrix $([X_i, Y_j])_{1 \leq i, j \leq d}$ is a non-zero polynomial defined over the commutator ideal $[\mathfrak{a}, \mathfrak{b}]$. For example, notice that if

$$\dim \mathfrak{a} = \dim \mathfrak{b} = \dim \mathfrak{z} = 1$$

then N is the 3-dimensional Heisenberg group. Also, our definition of bandlimitation is similar to the classical one, in the sense that a function is bandlimited if its group Fourier transform is supported on a bounded subset of the unitary dual of N .

First, we provide a precise description of a class of multiplicity-free bandlimited subspaces $\mathbf{H} \subset L^2(N)$, and an explicit construction of a single function $\phi \in \mathbf{H}$ such that if L is the left regular representation of N , there exists a set of the type

$$\Gamma = \prod_{k=1}^{n-2d} \exp(\mathbb{Z}Z_k) \prod_{k=1}^d \exp(\mathbb{Z}Y_k) \prod_{k=1}^d \exp(\mathbb{Z}X_k) \subset N$$

such that $L(\Gamma)\phi$ forms an orthonormal basis in \mathbf{H} , and is also a continuous wavelet for the Left regular representation restricted to \mathbf{H} . We also show that if the Lie algebra \mathfrak{n} has a rational structure then Γ is in fact a lattice subgroup of N . We organize the paper as follows. The second section deals with some preliminary facts which can also be found in the monograph [6], in [2], and [12]. The third section is the main part of the paper where we provide our results. In fact the following theorem summarizes the main result.

Theorem 1. *Let N be a simply connected, connected nilpotent Lie group with Lie algebra \mathfrak{n} satisfying the following. $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{b}, \mathfrak{b}] = \{0\}$, $\dim_{\mathbb{R}} \mathfrak{a} = \dim_{\mathbb{R}} \mathfrak{b} = d$, and assume that $\det([X_i, Y_j])_{1 \leq i, j \leq d}$ is a non-zero homogeneous polynomial over $[\mathfrak{a}, \mathfrak{b}]$. Moreover, let $\mathfrak{a} = \mathbb{R}\text{-span} \{X_1, \dots, X_d\}$, $\mathfrak{b} = \mathbb{R}\text{-span} \{Y_1, \dots, Y_d\}$, and $\mathfrak{z} = \mathbb{R}\text{-span} \{Z_1, \dots, Z_{n-2d}\}$. There is a basis for \mathfrak{n} , a closed left-invariant, bandlimited Hilbert space $\mathbf{H} \subset L^2(N)$, and a function $\phi \in \mathbf{H}$ such that if L is the left regular*

representation of N , the system $L(\Gamma)\phi$ forms an orthonormal basis for \mathbf{H} . Thus the Hilbert space \mathbf{H} is a sampling space with interpolation property. Moreover, ϕ is a Sinc-type function.

We also derive a precise formula for ϕ , and several examples are computed at the end of the paper.

2. PRELIMINARIES

Let us start by setting up some notations. In this paper, all representations are strongly continuous and unitary. All sets are measurable, and given two isomorphic representations τ and π , we write $\tau \cong \pi$. We also use the same notation for isomorphic Hilbert spaces. The characteristic function of a set E is written as χ_E , and the cardinal number of a countable set I is denoted by $\text{card}(I)$.

Let G be a locally compact group, and let Γ be a discrete subset of G . Let \mathbf{H} be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions. We call \mathbf{H} a **sampling space** with respect to Γ if the following properties hold. First, the mapping $R_\Gamma : \mathbf{H} \rightarrow l^2(\Gamma)$, $R_\Gamma f = (f(\gamma))_{\gamma \in \Gamma}$ is a scalar isometry. In other words, there exists a constant $c > 0$ such that for all $f \in \mathbf{H}$, $\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = c \|f\|_{\mathbf{H}}^2$. Secondly, there exists a vector $S \in \mathbf{H}$ such that for any vector $f \in \mathbf{H}$, we have the following expansion $f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1}x)$ with convergence in the L^2 -norm of \mathbf{H} . The function S is called a **sinc-type** function, and if R_Γ is surjective, we say that the sampling space \mathbf{H} has the **interpolation property**. In other words, every element of $l^2(\Gamma)$ can be interpolated by a function in \mathbf{H} .

Definition 2. Let (π, \mathbf{H}_π) denote a strongly continuous unitary representation of a locally compact group G . We say that the representation (π, \mathbf{H}_π) is **admissible** iff the map $W_\phi : \mathbf{H} \rightarrow L^2(G)$, $W_\phi \psi(x) = \langle \psi, \pi(x)\phi \rangle$ defines an isometry of \mathbf{H} into $L^2(G)$, and we say that ϕ is an **admissible vector** or a **continuous wavelet**.

It is known that if π is the left regular representation of G , and if G is connected and type I, then π is admissible if and only if G is nonunimodular.

Proposition 3. Let ϕ be an admissible vector for (π, \mathbf{H}_π) such that $\pi(\Gamma)\phi$ is a tight frame with frame constant c . Then $\mathbf{K} = W_\phi(\mathbf{H}_\pi)$ is a sampling space, and $W_\phi(\phi)$ is the associated **sinc-type** function for \mathbf{K} .

Proof. See Proposition 2.54 in [6]. \square

Proposition 4. *Let G be a unimodular group. Let \mathbf{H} be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions such that the mapping $f \mapsto (f(\gamma))_{\gamma \in \Gamma}$ is a scalar isometry. In other words, there exists a constant $c > 0$ such that for all $f \in H$, $\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = c \|f\|_{\mathbf{H}}^2$. Then \mathbf{H} is a sampling space. More precisely, there exists a unique self-adjoint convolution idempotent vector S such that $\frac{1}{c}S$ is the associated **sinc-type** function. In particular, for any function $f \in \mathbf{H}$ we have $f(\gamma) = \langle f, L(\gamma)S \rangle$. Moreover, \mathbf{H} has the interpolation property iff $L(\Gamma)S$ is an orthonormal basis of \mathbf{H} .*

Proof. See the Monograph [6]. \square

We will need to be familiar with the following facts in the third section.

Given a countable sequence $\{f_i\}_{i \in I}$ of functions in an Hilbert space \mathbf{H} , we say $\{f_i\}_{i \in I}$ forms a **frame** if and only if there exists strictly positive real numbers A, B such that for any function $f \in \mathbf{H}$,

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2.$$

In the case where $A = B$, the sequence of functions $\{f_i\}_{i \in I}$ forms a **tight frame**, and if $A = B = 1$, $\{f_i\}_{i \in I}$ is called a **Parseval frame**. Also, if $\{f_i\}_{i \in I}$ is a Parseval frame such that for all $i \in I$, $\|f_i\| = 1$ then $\{f_i\}_{i \in I}$ is an orthonormal basis for \mathbf{H} . A lattice Λ in \mathbb{R}^{2d} is a discrete additive subgroup of \mathbb{R}^{2d} . In other words, every point in Λ is isolated.

Let $\Lambda = A\mathbb{Z}^{2d}$ for some matrix A . We say Λ is a full rank lattice if A is non-singular, and we denote the dual of Λ by $\Lambda^\top = A^{-1tr}\Lambda$. We say a lattice is separable if $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. A **fundamental domain** D for a lattice in \mathbb{R}^d is a measurable set such that the followings hold, $(D+l) \cap (D+l') \neq \emptyset$ for distinct l, l' in Λ , and $\mathbb{R}^d = \bigcup_{l \in \Lambda} (D+l)$. We say D is a **packing set** for Λ if $\sum_{l \in \Lambda} \chi_D(x-l) \leq 1$ for almost every x or equivalently if $(D+l) \cap (D+l')$ has Lebesgue measure zero for any $l \neq l'$. Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full rank lattice in \mathbb{R}^{2d} and $g \in L^2(\mathbb{R}^d)$. The family of functions in $L^2(\mathbb{R}^d)$,

$$\mathcal{G}(f, A\mathbb{Z}^d \times B\mathbb{Z}^d) = \left\{ \begin{array}{l} e^{2\pi i \langle k, x \rangle} f(x-n) : \\ k \in B\mathbb{Z}^d, n \in A\mathbb{Z}^d \end{array} \right\}$$

is called a **Gabor system**. Let m be the Lebesgue measure on \mathbb{R}^d , and consider a full rank lattice $\Lambda = A\mathbb{Z}^d$ inside \mathbb{R}^d . The **volume** of Λ is defined as $\text{vol}(\Lambda) = m(\mathbb{R}^d/\Lambda) = |\det A|$. The **density** of Λ is defined as $d(\Lambda) = |\det A|^{-1}$.

Lemma 5. *Given a separable full-rank lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ in \mathbb{R}^{2d} . The following statements are equivalent.*

- (1) There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a Parseval frame in $L^2(\mathbb{R}^d)$.
- (2) $\text{vol}(\Lambda) = |\det A \det B| \leq 1$.
- (3) There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is complete in $L^2(\mathbb{R}^d)$.

Proof. See theorem 3.3 in [2]. □

Lemma 6. *Let Λ be a full rank lattice in \mathbb{R}^2 . There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, \Lambda)$ is an orthonormal basis if and only if $\text{vol}(\Lambda) = 1$. Also, if $\mathcal{G}(f, \Lambda)$ is a Parseval frame for $L^2(\mathbb{R}^d)$, then $\|f\|^2 = \text{vol}(\Lambda)$.*

Proof. See [2]. □

3. SAMPLING SPACES ON NILPOTENT LIE GROUPS

We will now restrict our attention to nilpotent Lie groups. Let N be a simply connected, connected nilpotent Lie group with Lie algebra \mathfrak{n} . The theory of sampling and interpolation is well understood when N is commutative. In the case where N is not commutative, the problem is much harder and several obstructions arise very quickly. For example, it is not clear what should be the substitute for \mathbb{Z} when \mathbb{R} is replaced with some general nilpotent Lie group N . In fact, there are many nilpotent Lie groups with no uniform lattice subgroups. Also, for general nilpotent Lie groups, the Fourier transform is an operator-valued transform, and the support set of the Plancherel measure is a non-trivial manifold contained in the dual of the Lie algebra (see [5]). However, there are some non-commutative nilpotent Lie groups which behave pretty well. For example the Heisenberg groups, and various generalizations or the Heisenberg groups.

Let N be a non-commutative connected, simply connected nilpotent Lie group with Lie algebra \mathfrak{n} over the reals. Fix a Jordan Hölder basis for \mathfrak{n} . Define $\mathfrak{a} = \mathbb{R}\text{-span} \{X_1, \dots, X_d\}$, $\mathfrak{b} = \mathbb{R}\text{-span} \{Y_1, \dots, Y_d\}$ such that the following assumptions hold.

- (1) $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{z}$, $\mathfrak{a}, \mathfrak{b}$ are commutative algebras, and $\dim_{\mathbb{R}}(\mathfrak{a}) = \dim_{\mathbb{R}}(\mathfrak{b}) = d$.
- (2) $\det \left(([X_i, Y_j])_{1 \leq i, j \leq d} \right)$ is a polynomial defined over the commutator ideal $[\mathfrak{n}, \mathfrak{n}]$.

This class of groups is a fairly large class of two-step nilpotent Lie groups containing the Heisenberg groups. We give below some examples of groups satisfying condition 1 and condition 2 given above.

Example 7. Let N be a nilpotent Lie group with Lie algebra \mathfrak{n} spanned by the Jordan Hölder basis $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ such that we have for non-trivial Lie brackets $[X_1, Y_1] = Z_1, [X_2, Y_1] = -Z_2, [X_1, Y_2] = Z_2, [X_2, Y_2] = Z_1$. Here is a faithful matrix representation of this nilpotent Lie group. We define the group homomorphism $\pi : N \rightarrow GL_7(\mathbb{R})$, and put

$$p = \exp z_1 Z_1 \exp z_2 Z_2 \exp y_1 Y_1 \exp y_2 Y_2 \exp x_1 X_1 \exp x_2 X_2.$$

The image of p under the representation π is the following matrix:

$$\begin{bmatrix} 1 & 0 & x_1 & x_2 & -y_1 & -y_2 & z_1 \\ 0 & 1 & -x_2 & x_1 & -y_2 & y_1 & z_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 8. Fix two natural numbers n and d , such that $n - 2d > 0$. Let M be a matrix of order d with entries in $\mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d}$ such that $\det(M)$ is a non-vanishing polynomial on $\mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d}$. Now let $\mathfrak{a} = \mathbb{R}\text{-span} \{X_1, \dots, X_d\}$, and $\mathfrak{b} = \mathbb{R}\text{-span} \{Y_1, \dots, Y_d\}$ such that $[X_i, Y_j] = M_{i,j}$. The Lie algebra

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus (\mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d})$$

satisfies all the conditions above. In fact this example exhausts all elements in the class of groups considered in this paper..

Proposition 9. Let \mathfrak{n} be a Lie algebra over the reals satisfying the following conditions (1) and (2) given above. The following facts hold.

- (1) $\exp(\mathfrak{n})$ is a non-commutative step-two nilpotent Lie group which is square-integrable modulo the center.

(2) Let $\lambda \in \mathfrak{n}^*$ and define

$$(2) \quad B(\lambda) = (\lambda[X_i, Y_j])_{1 \leq i, j \leq d}.$$

Then the unitary dual of $\exp \mathfrak{n}$ is parametrized by the smooth manifold

$$\Sigma = \left\{ \begin{array}{l} \lambda \in \mathfrak{n}^* : \det(B(\lambda)) \neq 0, \\ \lambda(X_1) = \cdots = \lambda(X_d) = 0, \\ \lambda(Y_1) = \cdots = \lambda(Y_d) = 0 \end{array} \right\}$$

which can be identified with a Zariski open subset of \mathbb{R}^{n-2d} .

(3) Let $d\lambda$ is the Lebesgue measure on Σ . The Plancherel measure of N is supported on Σ and is equal to

$$|\det B(\lambda)| d\lambda.$$

(4) The unitary dual of N which we denote by \widehat{N} is up to a null set equal to

$$\left\{ \begin{array}{l} \pi_\lambda = \text{Ind}_{\exp(\mathfrak{j} \oplus \mathfrak{b})}^{\exp \mathfrak{n}}(\chi_\lambda) : \lambda \in \Sigma \\ \text{where } \chi_\lambda(\exp X) = e^{-2\pi i \lambda(X)} \end{array} \right\}.$$

(5) Defining

$$Z = \sum_{i=1}^{n-2d} z_i Z_i, Y = \sum_{i=1}^d y_i Y_i, \text{ and } X = \sum_{i=1}^d x_i X_i,$$

we realize the representation π_λ as acting in $L^2(\mathbb{R}^d)$ such that

$$\pi_\lambda(\exp Z) f(t) = e^{2\pi i \lambda(Z)} f(t),$$

$$\pi_\lambda(\exp Y) f(t) = e^{-2\pi i \langle B(\lambda)y, t \rangle} f(t),$$

$$\text{and } \pi_\lambda(\exp X) f(t) = f(t - x).$$

Proof. The results in this proposition are some elementary facts in Harmonic Analysis of nilpotent Lie groups. See [8] where we specialized to the class of groups considered here. For more general nilpotent Lie groups we refer the interested reader to the standard reference [3]. \square

Definition 10. For a fixed $\lambda \in \Sigma$, let \mathfrak{u} be a lattice in $\mathfrak{a} \cong \mathbb{R}^d$ and let \mathfrak{v} be a lattice in $\mathfrak{b} \cong \mathbb{R}^d$. It is clear that for $f \in L^2(\mathbb{R}^d)$, $\pi_\lambda(\exp \mathfrak{u} \exp \mathfrak{v}) f$ is a Gabor system in $L^2(\mathbb{R}^d)$. We say that

$$\{\pi_\lambda(\exp \mathfrak{u} \exp \mathfrak{v}) f : \lambda \in \Sigma\}$$

is a **measurable field** of Gabor systems.

To make the paper more self-contained, we revisit the Plancherel theory for the class of groups considered in this paper. We fix a Jordan Hölder basis for the Lie algebra of \mathfrak{n} . Assume that N is endowed with its canonical Haar measure. \mathcal{P} denotes the Plancherel transform on $L^2(N)$, $\lambda = (\lambda_1, \dots, \lambda_{n-2d}, 0, \dots, 0) \in \Sigma$,

$$d\mu(\lambda) = |\det(B(\lambda))| d\lambda$$

is the Plancherel measure (see chapter 4 in [3]), and the matrix $B(\lambda)$ is defined in (2). We have

$$\mathcal{P} : L^2(N) \rightarrow \int_{\Sigma}^{\oplus} L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) d\mu(\lambda),$$

where the Fourier transform is defined on $L^2(N) \cap L^1(N)$ by

$$\mathcal{F}(f)(\lambda) = \int_{\Sigma} \pi_{\lambda}(n) f(n) dn,$$

and the Plancherel transform is the extension of the Fourier transform to $L^2(N)$ inducing the equality

$$\|f\|_{L^2(N)}^2 = \int_{\Sigma} \|\mathcal{P}(f)(\lambda)\|_{\mathcal{HS}}^2 d\mu(\lambda).$$

In fact, $\|\cdot\|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt norm on $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$. The inner product of arbitrary rank-one operators in $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ is given here.

$$\langle u \otimes v, w \otimes y \rangle_{\mathcal{HS}} = \langle u, w \rangle_{L^2(\mathbb{R}^d)} \langle v, y \rangle_{L^2(\mathbb{R}^d)}.$$

Let L be the left regular representation of the group N . It is well-known that

$$L \cong \mathcal{P} L \mathcal{P}^{-1} = \int_{\Sigma}^{\oplus} \pi_{\lambda} \otimes \mathbf{1}_{L^2(\mathbb{R}^d)} d\mu(\lambda),$$

and $\mathbf{1}_{L^2(\mathbb{R}^d)}$ is the identity operator on $L^2(\mathbb{R}^d)$. Finally, for $\lambda \in \Sigma$,

$$\mathcal{P}(L(x)\phi)(\lambda) = \pi_{\lambda}(x) \circ (\mathcal{P}\phi)(\lambda).$$

In order to proceed, we will also need the following facts.

Definition 11. We say a function $f \in L^2(N)$ is **bandlimited** if its Plancherel transform is supported on a bounded measurable subset of Σ . Fix

$$\mathbf{e} = \{\mathbf{e}(\lambda) : \lambda \in \Sigma\}$$

a measurable field of unit vectors in $L^2(\mathbb{R}^d)$. We say a Hilbert space is a **multiplicity-free** left-invariant closed subspace of $L^2(N)$ if and only if

$$\mathbf{H}(\mathbf{e}) = \mathcal{P}^{-1} \left(\int_{\Sigma}^{\oplus} L^2(\mathbb{R}^d) \otimes \mathbf{e}(\lambda) d\mu(\lambda) \right).$$

We define $\Gamma_1 = \exp(\mathbb{Z}Y_1 + \cdots + \mathbb{Z}Y_d) \exp(\mathbb{Z}X_1 + \cdots + \mathbb{Z}X_d)$, and

$$(3) \quad \Gamma = \exp(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_{n-2d}) \Gamma_1 \subset N$$

Lemma 12. *If \mathfrak{n} has a **rational structure** $[X_i, Y_j] = \sum_{k=1}^{n-2d} c_{ijk} Z_k$, $c_{ijk} \in \mathbb{Q}$ for all $1 \leq i, j \leq d$) then Γ is a lattice subgroup of N .*

Proof. Let $Z(k) = k_1 Z_1 + \cdots + k_{n-2d} Z_{n-2d}$, $Y(m) = m_1 Y_1 + \cdots + m_d Y_d$ and $X(l) = l_1 X_1 + \cdots + l_d X_d$. There exists $M \in GL(n-2d, \mathbb{Q})$ such that $\{Z(k) + \frac{1}{2}[Y(m), X(l)] : k_i, m_j, l_i \in \mathbb{Z}\}$ is isomorphic to $M\mathbb{Z}^{n-2d}$ which we set to be equal to \mathfrak{d} . Thanks to the Baker-Campbell-Hausdorff formula,

$$\Gamma = \exp \left(\mathfrak{d} + \sum_{j=1}^d \mathbb{Z}Y_j + \sum_{k=1}^d \mathbb{Z}X_k \right).$$

Thus, Γ is a lattice subgroup of N . For the second part, to show that Γ is a non-type I group, it suffices to show that Γ contains no abelian subgroup of finite index (see Theorem 7.8 in [5]). Suppose that there exists $\Gamma' \leq \Gamma$ such that Γ' is abelian and $\text{card}(\Gamma/\Gamma')$ is finite. Clearly $\Gamma' \neq \Gamma$ since Γ is not abelian. Now pick $\gamma \in \Gamma$ such that γ is a non trivial element and $\gamma\Gamma' \neq \Gamma'$. Thus, $\gamma\Gamma' \in \Gamma/\Gamma'$. There exists $X \in \mathfrak{n}$ such that $\gamma = \exp X$ and of course $\gamma^j = \exp(jX) \notin \Gamma'$. The group generated by the element $\gamma\Gamma'$ is isomorphic to \mathbb{Z} and is contained in Γ/Γ' . We have reached a contradiction. Thus, Γ is a non-type I group. This completes the proof. \square

Let

$$\mathbf{E} = \Sigma \cap \{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\}.$$

We fix $\mathbf{e} = \{\mathbf{e}_\lambda : \lambda \in \mathbf{E}\}$ a measurable field of unit vectors in $L^2(\mathbb{R}^d)$. Let $C \subset \mathfrak{z}^*$ be a compact set such that $\{e^{2\pi i \langle k, \lambda \rangle} \chi_{\mathbf{E} \cap C}(\lambda) : \lambda \in \mathbf{E} \cap C\}$ is a Parseval frame for $L^2(\mathbf{E} \cap C, d\lambda)$. We define the multiplicity-free closed left-invariant Hilbert subspace $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$ as follows.

$$(4) \quad \mathbf{H}(\mathbf{e}, \mathbf{E} \cap C) = \mathcal{P}^{-1} \left(\int_{\mathbf{E} \cap C}^{\oplus} L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda |\det B(\lambda)| d\lambda \right).$$

Lemma 13. *Let $\phi \in L^2(N)$. If for a.e. $\lambda \in \mathbf{E} \cap C$, $T_\lambda = \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda$, $\mathcal{P}\phi(\lambda) = |\det B(\lambda)|^{-1/2} T_\lambda$, and if the system $\mathcal{G}(\mathbf{u}_\lambda, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$, then $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$.*

Proof. See [8]. □

Here is a sufficient condition to obtain an orthonormal basis of the type $L(\Gamma)\phi$ in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$.

Lemma 14. *If $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$ as described in Lemma (13) and if $\int_{\mathbf{E} \cap C} |\det B(\lambda)| d\lambda = 1$ then $L(\Gamma)\phi$ is an **orthonormal basis**.*

Proof. It suffices to show that $\|\phi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)} = 1$. Recall from Lemma 13 that for a.e. $\lambda \in \Sigma$, $T_\lambda = \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda$, if $\mathcal{P}\phi(\lambda) = |\det B(\lambda)|^{-1/2} T_\lambda$, and if the system $\mathcal{G}(\mathbf{u}_\lambda, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d)$ forms a Parseval frame in $L^2(\mathbb{R}^d)$ then $\|\mathbf{u}_\lambda\|_{L^2(\mathbb{R}^d)}^2 = |\det B(\lambda)|$ a.e. Now,

$$\begin{aligned} \|\phi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)}^2 &= \int_{\mathbf{E} \cap C} \|\mathcal{P}\phi(\lambda)\|_{\mathcal{HS}}^2 |\det B(\lambda)| d\lambda \\ &= \int_{\mathbf{E} \cap C} \|\mathbf{u}_\lambda\|_{L^2(\mathbb{R}^d)}^2 d\lambda \\ &= \int_{\mathbf{E} \cap C} |\det B(\lambda)| d\lambda \\ &= 1. \end{aligned}$$

Since any unit-norm Parseval frame is an orthonormal basis, the proof is completed. □

Notice that Lemma 14 does not guarantee the existence of an orthonormal basis of the type $L(\Gamma)\phi$. In order to proceed, we will have to answer the following question.

Question 1. *Is it possible for the following two conditions to be both true? Firstly, there exists $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$ such that $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$. Secondly, $\int_{\mathbf{E} \cap C} |\det B(\lambda)| d\lambda = 1$.*

The answer to the question is yes. However, the proof will require some non-trivial effort. The remaining of this paper will be devoted to providing some clear answers to the question. Here is a general idea of how we proceed. We first decompose the Hilbert

space $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)$ into a finite direct sum of some ‘special’ Hilbert subspaces admitting Parseval frame generators as follows:

$$\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C) = \bigoplus_{j=1}^m \mathbf{H}_j.$$

We construct a Parseval frame generator for each Hilbert subspace \mathbf{H}_j , and we sum the Parseval frames generators to obtain an orthonormal basis.

Lemma 15. *The basis elements of \mathbf{n} can be rescaled so that*

$$\mu(\{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\}) \geq 1.$$

Proof. Let $\mathbf{E} = \{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\}$. If $\mu(\mathbf{E}) > 1$ then we are done. So let us suppose that

$$\mu(\mathbf{E}) = \int_{\mathbf{E}} |\det B(\lambda)| d\lambda < 1.$$

Pick $\alpha > 1$. Put $B(\lambda) = (\lambda[X_i, Y_j])_{1 \leq i, j \leq d}$ and

$$B'(\lambda) = (\lambda[\alpha X_i, Y_j])_{1 \leq i, j \leq d}.$$

It follows that $\det B'(\lambda) = \alpha \det B(\lambda)$. Now let

$$\mathbf{E}' = \{\lambda \in \Sigma : |\det B'(\lambda)| \leq 1\}.$$

Notice that because $\alpha > 1$, $\mathbf{E}' \subseteq \mathbf{E}$ and $\int_{\mathbf{E}'} |\det B(\lambda)| d\lambda$ is finite since $\int_{\mathbf{E}'} |\det B(\lambda)| d\lambda \leq \int_{\mathbf{E}} |\det B(\lambda)| d\lambda$. Put

$$d\mu'(\lambda) = |\det B'(\lambda)| d\lambda$$

and pick α large enough so that

$$\mu'(\mathbf{E}') = \alpha \int_{\mathbf{E}'} |\det B(\lambda)| d\lambda \geq 1.$$

Our new basis is then

$$\{Z_1, \dots, Z_{n-2d}, Y_1, \dots, Y_d, \alpha X_1, \dots, \alpha X_d\}.$$

□

From now on, we will assume that a choice of basis has been made so that the condition

$$\mu(\{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\}) \geq 1$$

holds. Thus, there exists a compact subset \mathbf{C} of \mathfrak{z}^* such that

$$\mu(\{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\} \cap \mathbf{C}) = 1.$$

Put

$$\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}) = \mathcal{P}^{-1} \left(\int_{\mathbf{E} \cap \mathbf{C}}^{\oplus} L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda |\det B(\lambda)| d\lambda \right).$$

Notice that the first condition described in Question 1 is easily obtained for an appropriate choice of a basis for the Lie algebra \mathfrak{n} . However, the first condition is hard to obtain simultaneously with the second.

Let $\iota : \mathbb{R}^{n-2d} \rightarrow \mathfrak{z}$ defined by

$$\iota(\lambda_1, \dots, \lambda_{n-2d}) = (\lambda_1, \dots, \lambda_{n-2d}, 0, \dots, 0).$$

Lemma 16. *If $\mu(\{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\}) \geq 1$, then there exists a finite partition of $\mathbf{E} \cap \mathbf{C} = \{\lambda \in \Sigma : |\det B(\lambda)| \leq 1\} \cap \mathbf{C}$ denoted \mathbf{P} such that*

- (1) $\mathbf{E} \cap \mathbf{C} = \bigcup_{A^j \in \mathbf{P}} A_j$.
- (2) $\iota^{-1}(A_j)$ is contained in a fundamental domain of $\mathbb{R}^{n-2d}/\mathbb{Z}^{n-2d}$.
- (3) For each j , where $1 \leq j \leq \text{card}(\mathbf{P})$, there exists a Parseval frame (not necessarily an orthonormal basis) of the type $L(\Gamma)\phi_j$ for

$$\mathbf{H}_j(\mathbf{e}, \mathbf{E} \cap \mathbf{C}) = \mathcal{P}^{-1} \left(\int_{\mathbf{E} \cap \mathbf{C} \cap A_j}^{\oplus} L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda |\det B(\lambda)| d\lambda \right).$$

- (4) For each j , satisfying $1 \leq j \leq \text{card}(\mathbf{P})$, we have

$$\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}) = \bigoplus_{j=1}^{\text{card}(\mathbf{P})} \mathbf{H}_j(\mathbf{e}, \mathbf{E} \cap \mathbf{C}).$$

Proof. The proofs for Parts 1, 2 are obvious. The proof for Part 3 follows from Lemma 13, and Part 4 follows from Parts 1, 2 and 3. \square

Lemma 17. *For each $1 \leq j \leq \text{card}(\mathbf{P})$, we can construct a Parseval frame of the type $L(\Gamma)\phi_j$, such that*

$$\left\| \sum_{j=1}^{\text{card}(\mathbf{P})} \phi_j \right\|_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})}^2 = 1.$$

However, in general $\sum_{j=1}^{\text{card}(\mathbf{P})} L(\Gamma)\phi_j$ is **not** a Parseval frame for $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$.

Proof. Let $\mathbf{E}^\circ = \mathbf{E} \cap \mathbf{C}$. The existence of a Parseval frame for each $\mathbf{H}_j(\mathbf{e}, \mathbf{E}^\circ)$, $1 \leq j \leq \text{card}(\mathbf{P})$ of the type $L(\Gamma)\phi_j$ is given in [8], and

$$\begin{aligned} \left\| \sum_{j=1}^{\text{card}(\mathbf{P})} \phi_j \right\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2 &= \sum_{j=1}^{\text{card}(\mathbf{P})} \|\phi_j\|_{\mathbf{H}_j(\mathbf{e}, \mathbf{E}^\circ)}^2 \\ &= \int_{\cup_{j=1}^{\text{card}(\mathbf{P})} (\mathbf{E}^\circ \cap A_j)} |\det B(\lambda)| d\lambda. \end{aligned}$$

Since, $\cup_{j=1}^{\text{card}(\mathbf{P})} (\mathbf{E}^\circ \cap A_j) = \mathbf{E}^\circ$ then

$$\left\| \sum_{j=1}^{\text{card}(\mathbf{P})} \phi_j \right\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2 = \int_{\mathbf{E}^\circ} |\det B(\lambda)| d\lambda = 1.$$

Finally, for the second part of the proof, it is well-known that in general the direct sum of Parseval frames is not a Parseval frame. \square

From now on, we endow \mathbb{R}^d with its **max norm**. In other words, given $x \in \mathbb{R}^d$, if $x = (x_1, \dots, x_d)$, $\|x\| = \max_{1 \leq i \leq d} |x_i|$. The **unit ball** in \mathbb{R}^d is the set of all vectors x satisfying the condition $\|x\| < 1$.

Lemma 18. *Let Ω be a fundamental domain for \mathbb{Z}^d strictly contained in the unit ball of \mathbb{R}^d such that $\Omega - \Omega$ is also contained in the unit ball of \mathbb{R}^d . If $\|B(\lambda)^{tr}\| < 1$ for a.e $\lambda \in \mathbf{E} \cap \mathbf{C}$ then Ω is a packing set for $B(\lambda)^\top \mathbb{Z}^d$ for a.e $\lambda \in \mathbf{E} \cap \mathbf{C}$.*

Proof. By contradiction suppose that $\|B(\lambda)^{tr}\| < 1$ and there exists a set of positive measure $U \subset \mathbf{E} \cap \mathbf{C}$ such that for any $\lambda \in S$, there exists $k, l \in \mathbb{Z}^d$ satisfying $(\Omega + B(\lambda)^\top k) \cap (\Omega + B(\lambda)^\top l) \neq \emptyset$ for $l \neq k$. Put $p = l - k$. There exist $x, y \in \Omega$ such that $x + B(\lambda)^\top k = y + B(\lambda)^\top l$. Thus, $B(\lambda)^{tr}(x - y) = l - k$ and $\|p\| = \|B(\lambda)^{tr}(x - y)\|$. Since p is a non-trivial element of \mathbb{Z}^d ,

$$1 \leq \|B(\lambda)^{tr}(x - y)\| \leq \|B(\lambda)^{tr}\| \|x - y\|,$$

and

$$\|B(\lambda)^{tr}\| \|x - y\| \leq \|B(\lambda)^{tr}\| \left(\sup_{z \in \Omega - \Omega} \|z\| \right) < \sup_{z \in \Omega - \Omega} \|z\| \leq 1 \text{ on } U.$$

So, $1 \leq \|B(\lambda)^{tr}\| \leq 1$ on S and $\|B(\lambda)^{tr}\| = 1$ on S . That would be a contradiction. \square

Corollary 19. *If $\|B(\lambda)^{tr}\| < 1$ for a.e. $\lambda \in \mathbf{E} \cap \mathbf{C}$ then $[-1/2, 1/2]^d$ is a packing set for $(B(\lambda)^{tr})^{-1} \mathbb{Z}^d$, for a.e. $\lambda \in \mathbf{E} \cap \mathbf{C}$.*

Lemma 20. *If $\|B(\lambda)^{tr}\| < 1$ for a.e. $\lambda \in \mathbf{E} \cap \mathbf{C}$, then*

$$\mathcal{G}\left(\chi_{[-1/2, 1/2]^d}, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d\right)$$

is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $1/|\det B(\lambda)|$.

Proof. By Proposition 3.1 in [12], the Lemma is true. \square

Lemma 21. *There exists a basis for the Lie algebra of N for which $\mathbf{E} \cap \mathbf{C} \subset \Sigma$ such that for all $\lambda \in \mathbf{E} \cap \mathbf{C}$, $\|B(\lambda)^{tr}\| < 1$.*

Proof. Let $\mathbf{I} = \{\lambda \in \Sigma : |\det B(\lambda)| < 1\}$, and let

$$\mathbf{R} = \{\lambda \in \Sigma : \|B(\lambda)^{tr}\| < 1\}.$$

Notice that both sets are open in Σ and $\mu(\mathbf{I} \cap \mathbf{R}) > 0$. We have two separate cases to treat. For the first case, let us assume that

$$\mu(\mathbf{I} \cap \mathbf{R}) \geq 1.$$

There exists a compact set $\mathbf{C} \subset \mathfrak{z}^*$ such that $\mu(\mathbf{I} \cap \mathbf{R} \cap \mathbf{C}) = 1$. For the second case, assume that $\mu(\mathbf{I} \cap \mathbf{R}) \in (0, 1)$. We pick a new basis

$$\{Z_1, \dots, Z_{n-2d}, X'_1, \dots, X'_d, Y'_1, \dots, Y'_d\}$$

for the Lie algebra \mathfrak{n} such that $B_\beta(\lambda) = (\lambda[X'_i, Y'_j])_{1 \leq i, j \leq d}$ and $B_\beta(\lambda) = \beta^{-1}(B(\lambda))$ for a.e. λ and $\beta > 1$. Next, we define

$$\mathbf{I}_\beta = \{\lambda \in \Sigma : |\det B_\beta(\lambda)| < 1\} = \{\lambda \in \Sigma : |\det B(\lambda)| < \beta^d\},$$

$$\mathbf{R}_\beta = \{\lambda \in \Sigma : \|B_\beta(\lambda)^{tr}\| < 1\} = \{\lambda \in \Sigma : \|B(\lambda)^{tr}\| < \beta\}.$$

First notice that $\mathbf{I}_\beta \cap \mathbf{R}_\beta \supset \mathbf{I} \cap \mathbf{R}$, and also that as $\beta \rightarrow \infty$, $\mathbf{I}_\beta \cap \mathbf{R}_\beta \rightarrow \Sigma$. Thus $\mu(\mathbf{I}_\beta \cap \mathbf{R}_\beta) > \mu(\mathbf{I} \cap \mathbf{R})$ and for β large enough, we have

$$\mu(\mathbf{I}_\beta \cap \mathbf{R}_\beta) > 1.$$

We can then find a compact set $\mathbf{C}_\beta \subset \mathfrak{z}^*$ such that $\mu(\mathbf{I}_\beta \cap \mathbf{R}_\beta \cap \mathbf{C}_\beta) = 1$. \square

Theorem 22. *Suppose that for a.e. $\lambda \in \mathbf{E} \cap \mathbf{C}$, $\|B(\lambda)^{tr}\| < 1$. For each $1 \leq j \leq \text{card}(\mathbf{P})$, there exists $\phi_j \in \mathbf{H}_j(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such*

that $L(\Gamma)\phi_j$ is a Parseval frame for $\mathbf{H}_j(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$, and for $\phi = \sum_{j=1}^{\text{card}(\mathbf{P})} \phi_j$, $L(\Gamma)\phi$ is an **ONB** in

$$\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}) = \bigoplus_{j=1}^{\text{card}(\mathbf{P})} \mathbf{H}_j(\mathbf{e}, \mathbf{E} \cap \mathbf{C}).$$

Proof. Put $\mathbf{E}^\circ = \mathbf{E} \cap \mathbf{C}$. For $j = 1$, let us define the function $\phi_1 \in \mathbf{H}_1(\mathbf{e}, \mathbf{E}^\circ)$ such that

$$\mathcal{P}(\phi_1)(\lambda) = (\mathbf{u}_\lambda^1 \otimes \mathbf{e}_\lambda) |\det B(\lambda)|^{-1/2},$$

and the Gabor system $\mathcal{G}(\mathbf{u}_\lambda^1, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$ forms a Parseval Gabor frame in $L^2(\mathbb{R}^d)$. As shown in Cor(19) we can pick a measurable subset $E(\lambda)$ of \mathbb{R}^d such that $E(\lambda) = [-1/2, 1/2]^d$ is a fundamental domain for \mathbb{Z}^d and a packing set for $B(\lambda)^\top \mathbb{Z}^d$ for a.e. $\lambda \in \mathbf{E}^\circ$. By Lemma 20,

$$\mathcal{G}(\chi_{E(\lambda)}, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$$

is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $1/|\det B(\lambda)|$. Next, if we let $\mathbf{u}_\lambda^1 = |\det B(\lambda)|^{1/2} \chi_{E(\lambda)}$ then $\mathcal{G}(\mathbf{u}_\lambda^1, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$ forms a Gabor Parseval frame in $L^2(\mathbb{R}^d)$. We define for $j \neq 1$, $\phi_j \in \mathbf{H}_j(\mathbf{e}, \mathbf{E}^\circ)$ such that $\mathcal{P}\phi_j(\lambda) = (\mathbf{u}_\lambda^j \otimes \mathbf{e}_\lambda) |\det B(\lambda)|^{-1/2}$ and

$$(5) \quad \mathbf{u}_\lambda^j = |\det B(\lambda)|^{1/2} \chi_{E(\lambda)+k_j}, k_j \in \mathbb{Z}^d.$$

Furthermore, we assume that for each $1 \leq j, l \leq \text{card}(\mathbf{P})$, $k_j \neq k_l$ whenever $j \neq l$. Now, let ψ be any arbitrary element in

$$\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \bigoplus_{j=1}^{\text{card}(\mathbf{P})} \mathbf{H}_j(\mathbf{e}, \mathbf{E}^\circ)$$

such that $\psi = \sum_{j=1}^{\text{card}(\mathbf{P})} \psi_j$, for $\psi_j \in \mathbf{H}_j(\mathbf{e}, \mathbf{E}^\circ)$. Recall that

$$\Gamma_1 = \exp(\mathbb{Z}Y_1 + \cdots + \mathbb{Z}Y_d) \exp(\mathbb{Z}X_1 + \cdots + \mathbb{Z}X_d).$$

Next, letting $\phi = \sum_{j=1}^{\text{card}(\mathbf{P})} \phi_j$, and $r(\lambda) = |\det B(\lambda)|$,

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi \rangle|^2 = \sum_{\gamma \in \Gamma} \left| \int_{\mathbf{E}^\circ} \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma) \mathcal{P}\phi(\lambda) \rangle_{\mathcal{HS}} r(\lambda) d\lambda \right|^2.$$

Using the fact that $\pi_\lambda (\Gamma \cap \mathfrak{J}) \chi_{\mathbf{E}^\circ}$ forms a Parseval frame for $L^2 (\mathbf{E}^\circ)$, (see Part 2 of Lemma 16); defining functions $s_{\gamma_1}, s_{\gamma_1}^j$ on \mathbf{E}° by

$$s_{\gamma_1} (\lambda) = \left\langle \mathcal{P}\psi (\lambda), \pi_\lambda (\gamma_1) \mathcal{P}\phi (\lambda) r (\lambda)^{1/2} \right\rangle_{\mathcal{HS}}, \text{ and}$$

$$s_{\gamma_1}^j (\lambda) = \left\langle \mathcal{P}\psi_j (\lambda), \pi_\lambda (\gamma_1) \mathcal{P}\phi_j (\lambda) r (\lambda)^{1/2} \right\rangle_{\mathcal{HS}},$$

we have

$$(6) \quad \sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma) \phi \rangle|^2 = \int_{\mathbf{E}^\circ} \sum_{\gamma_1 \in \Gamma_1} |s_{\gamma_1} (\lambda)|^2 r (\lambda) d\lambda$$

$$= \int_{\mathbf{E}^\circ} \sum_{\gamma_1 \in \Gamma_1} \left| \sum_{j=1}^{\text{card}(\mathbf{P})} s_{\gamma_1}^j (\lambda) \right|^2 r (\lambda) d\lambda.$$

Now, letting $\mathcal{P}\psi_j (\lambda) = w_\lambda^j \otimes \mathbf{e}_\lambda$, we have

$$s_{\gamma_1}^j (\lambda) = \langle w_\lambda^j, \pi_\lambda (\gamma_1) \mathbf{u}_\lambda^j \rangle_{L^2(\mathbb{R}^d)}.$$

Thus, for $k, m \in \mathbb{Z}^d$,

$$s_{\gamma_1}^j (\lambda) = \int_{\mathbb{R}^d} w_\lambda^j (t) e^{2\pi i \langle k, B(\lambda)^{tr} t \rangle} r (\lambda)^{-1/2} \chi_{E(\lambda)+k_j} (t-m) dt$$

$$= \int_{B(\lambda)^{tr}(E(\lambda)+k_j+m)} w_\lambda^j (B(\lambda)^{-tr} t) e^{2\pi i \langle k, t \rangle} r (\lambda)^{-1/2} dt.$$

Since $E(\lambda)$ is a tiling set for \mathbb{R}^d , if

$$j, l \in \{1, \dots, \text{card}(\mathbf{P})\},$$

and if $j \neq l$ then

$$B(\lambda)^{tr} (E(\lambda) + k_j + m) \cap B(\lambda)^{tr} (E(\lambda) + k_l + m) = \emptyset.$$

Put $s_{\gamma_1}^j (\lambda) = s_{k,m}^j (\lambda)$. For each $m \in \mathbb{Z}^d$, the sequences

$$(s_{k,m}^j (\lambda))_{k \in \mathbb{Z}^d} \text{ and } (s_{k,m}^l (\lambda))_{k \in \mathbb{Z}^d}$$

are orthogonal because they are Fourier inverses of the following orthogonal functions:

$$w_\lambda^j (B(\lambda)^{-tr} t) r (\lambda)^{-1/2} \chi_{B(\lambda)^{tr}(E(\lambda)+k_j+m)} (t)$$

and

$$w_\lambda^l (B(\lambda)^{-tr} t) r (\lambda)^{-1/2} \chi_{B(\lambda)^{tr}(E(\lambda)+k_l+m)} (t).$$

Thus,

$$\sum_{\gamma_1 \in \Gamma_1} \left| \sum_{j=1}^{\text{card}(\mathbf{P})} s_{\gamma_1}^j(\lambda) \right|^2 = \sum_{\gamma_1 \in \Gamma_1} \sum_{j=1}^{\text{card}(\mathbf{P})} |s_{\gamma_1}^j(\lambda)|^2.$$

Coming back to Equation (6), we have

$$\begin{aligned} |\langle \psi, L(\gamma) \phi \rangle|^2 &= \int_{\mathbf{E}^\circ} \sum_{\gamma_1 \in \Gamma_1} \left| \sum_{j=1}^{\text{card}(\mathbf{P})} s_{\gamma_1}^j(\lambda) \right|^2 r(\lambda) d\lambda \\ &= \int_{\mathbf{E}^\circ} \sum_{j=1}^{\text{card}(\mathbf{P})} \sum_{\gamma_1 \in \Gamma_1} |s_{\gamma_1}^j(\lambda)|^2 r(\lambda) d\lambda, \end{aligned}$$

and

$$\begin{aligned} |\langle \psi, L(\gamma) \phi \rangle|^2 &= \sum_{j=1}^{\text{card}(\mathbf{P})} \int_{\mathbf{E}^\circ \cap A_j} (\|\mathcal{P}\psi_j(\lambda)\|_{\mathcal{HS}}^2) r(\lambda) d\lambda \\ &= \sum_{j=1}^{\text{card}(\mathbf{P})} \|\psi_j\|_{\mathbf{H}_j(\mathbf{e}, \mathbf{E}^\circ)}^2 \\ &= \|\psi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2. \end{aligned}$$

Now, we compute the norm of the vector ϕ .

$$\begin{aligned} \|\phi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})}^2 &= \int_{\mathbf{E}^\circ} \|\mathcal{P}\phi_j(\lambda)\|_{\mathcal{HS}}^2 r(\lambda) d\lambda \\ &= \int_{\mathbf{E}^\circ} \sum_{j=1}^{\text{card}(\mathbf{P})} \left\| (\mathbf{u}_\lambda^j \otimes \mathbf{e}_\lambda) r(\lambda)^{-1/2} \right\|_{\mathcal{HS}}^2 r(\lambda) d\lambda \\ &= \int_{\mathbf{E}^\circ} \sum_{j=1}^{\text{card}(\mathbf{P})} \|\mathbf{u}_\lambda^j \otimes \mathbf{e}_\lambda\|_{\mathcal{HS}}^2 d\lambda \\ &= \int_{\mathbf{E}^\circ} \sum_{j=1}^{\text{card}(\mathbf{P})} \|\mathbf{u}_\lambda^j\|_{L^2(\mathbb{R}^d)}^2 d\lambda. \end{aligned}$$

Using the fact that $\mathbf{E}^\circ = \bigcup_{j=1}^{\text{card}(\mathbf{P})} (\mathbf{E}^\circ \cap A_j)$,

$$\begin{aligned} \|\phi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2 &= \int_{\bigcup_{j=1}^{\text{card}(\mathbf{P})} (\mathbf{E}^\circ \cap A_j)} \|\mathbf{u}_\lambda^j\|_{L^2(\mathbb{R}^d)}^2 d\lambda \\ &= \int_{\mathbf{E}^\circ} |\det B(\lambda)| d\lambda = 1. \end{aligned}$$

Since L is a unitary operator, and since $L(\Gamma)\phi$ forms a unit norm Parseval frame, $L(\Gamma)\phi$ forms an orthonormal basis in $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$. \square

In summary, we fix a basis for the Lie algebra of N such that for all $\lambda \in \mathbf{E} \cap \mathbf{C} \subset \Sigma$, we have $\|B(\lambda)^{tr}\| < 1$. Defining $\phi = \sum_{j=1}^{\text{card}(\mathbf{P})} \phi_j$ such that

$$\mathcal{P}(\phi_k)(\lambda) = \left(\chi_{[-1/2, 1/2]^d + k} \otimes \chi_{[-1/2, 1/2]^d + k} \right),$$

it follows that $L(\Gamma)\phi$ is an orthonormal basis for the Hilbert space $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ where

$$\mathbf{e} = \left\{ \mathbf{e}(\lambda) = \chi_{[-1/2, 1/2]^d + k} : \lambda \in \mathbf{E} \cap \mathbf{C} \right\}.$$

Now, we fix a Gabor orthonormal basis

$$\mathcal{B} = \left\{ b_{l,j}(t) = e^{2\pi i \langle l, t \rangle} \chi_{[-1/2, 1/2]^d}(t - j) : (j, l) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\}$$

for $L^2(\mathbb{R}^d)$. We derive a formula for ϕ as follows. Let \mathbf{F} be the Fourier transform defined on $L^2(\mathbb{R}^d)$. Let $n \in N, \lambda \in \mathbf{E} \cap \mathbf{C}$, and $(j, l) \in \mathbb{Z}^d \times \mathbb{Z}^d$. Let $n = zyx$ where z is central element, $y \in \exp(\mathfrak{b})$ and $x \in \exp(\mathfrak{a})$. Put $E = [-1/2, 1/2]^d \subset \mathbb{R}^d$. We define $\mathbf{f}(n, \lambda, l, j) = \mathbf{f}(zyx, \lambda, l, j)$ as being equal to

$$e^{2\pi i \langle \lambda, z \rangle} e^{2\pi i \langle B(\lambda)y + l, j \rangle} \mathbf{F} \left(\chi_{E \cap E + k + x - j} e^{2\pi i \langle B(\lambda)y, \cdot \rangle} \right) (l).$$

Proposition 23. Put $d\mu(\lambda) = \det |B(\lambda)| d\lambda$. A sinc-type function in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ has the following precise formula:

(7)

$$\phi(n) = \int_{\mathbf{E} \cap \mathbf{C}} \sum_{\substack{(j,l) \in \mathbb{Z}^d \times \mathbb{Z}^d \\ k \in \{1, \dots, \text{card}(\mathbf{P})\}}} (\delta_{j,k} (e^{2\pi i \langle l, k \rangle} \text{sinc}(l)) \mathbf{f}(n, \lambda, l, j)) d\mu(\lambda).$$

Proof. Using the inversion formula provided in Theorem 4.4 in the monograph [6],

$$\phi(n) = \int_{\mathbf{E} \cap \mathbf{C}} \text{trace} \left(\sum_{k=1}^{\text{card}(\mathbf{P})} \mathcal{P}(\phi_k)(\lambda) \circ \pi_\lambda(n)^{-1} \right) \det |B(\lambda)| d\lambda.$$

Also,

$$\begin{aligned}
& \text{trace} \left(\sum_{k=1}^{\text{card}(\mathbf{P})} \mathcal{P}(\phi_k)(\lambda) \circ \pi_\lambda(n)^{-1} \right) \\
&= \sum_{(j,l) \in \mathbb{Z}^d \times \mathbb{Z}^d} \left\langle \sum_{k=1}^{\text{card}(\mathbf{P})} \left(\chi_{[-1/2, 1/2]^d + k} \otimes \chi_{[-1/2, 1/2]^d + k} \right) \pi_\lambda(n)^{-1} b_{l,j}, b_{l,j} \right\rangle \\
&= \sum_{(j,l) \in \mathbb{Z}^d \times \mathbb{Z}^d} \sum_{k=1}^{\text{card}(\mathbf{P})} \left\langle \left(\chi_{[-1/2, 1/2]^d + k} \otimes \chi_{[-1/2, 1/2]^d + k} \right) \pi_\lambda(n)^{-1} b_{l,j}, b_{l,j} \right\rangle \\
&= \sum_{(j,l) \in \mathbb{Z}^d \times \mathbb{Z}^d} \sum_{k=1}^{\text{card}(\mathbf{P})} \left\langle b_{l,j}, \chi_{[-1/2, 1/2]^d + k} \right\rangle \left\langle \pi_\lambda(n^{-1}) b_{l,j}, \chi_{[-1/2, 1/2]^d + k} \right\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
\left\langle b_{l,j}, \chi_{[-1/2, 1/2]^d + k} \right\rangle &= \int e^{2\pi i \langle l, t \rangle} \chi_{[-1/2, 1/2]^d}(t - j) \chi_{[-1/2, 1/2]^d + k}(t) dt \\
&= \delta_{j,k} \int_{[-1/2, 1/2]^d + k} e^{2\pi i \langle l, t \rangle} dt.
\end{aligned}$$

Let \mathbf{F} be the Fourier transform defined on $L^2(\mathbb{R}^d)$. Then

$$\left\langle b_{l,j}, \chi_{[-1/2, 1/2]^d + k} \right\rangle = \delta_{j,k} \mathbf{F} \left(\chi_{[-1/2, 1/2]^d + k} \right) (l) = \delta_{j,k} e^{2\pi i \langle l, k \rangle} \text{sinc}(l).$$

Recall that $n = zyx$ where z is central element, $y \in \exp(\mathfrak{b})$ and $x \in \exp(\mathfrak{a})$. Put $E = [-1/2, 1/2]^d$.

$$\begin{aligned}
\left\langle \pi_\lambda(n^{-1}) b_{l,j}, \chi_{E+k} \right\rangle &= \langle b_{l,j}, \pi_\lambda(n) \chi_{E+k} \rangle \\
&= e^{2\pi i \langle \lambda, z \rangle} \int e^{2\pi i \langle l, s+j \rangle} \chi_E(s) e^{2\pi i \langle B(\lambda)y, s+j \rangle} \chi_{E+k}(s+j-x) ds \\
&= e^{2\pi i \langle \lambda, z \rangle} \int e^{2\pi i \langle l, s+j \rangle} \chi_E(s) e^{2\pi i \langle B(\lambda)y, s+j \rangle} \chi_{E+k+x-j}(s) ds \\
&= e^{2\pi i \langle \lambda, z \rangle} e^{2\pi i \langle l, j \rangle} e^{2\pi i \langle B(\lambda)y, j \rangle} \int_{E \cap E+k+x-j} e^{2\pi i \langle l, s \rangle} e^{2\pi i \langle B(\lambda)y, s \rangle} ds \\
&= e^{2\pi i \langle \lambda, z \rangle} e^{2\pi i \langle B(\lambda)y, l+j \rangle} \mathbf{F} \left(\chi_{E \cap E+k+x-j}(\cdot) e^{2\pi i \langle B(\lambda)y, \cdot \rangle} \right) (l).
\end{aligned}$$

Finally, we obtain the formula in (7). \square

Proposition 24. *Let ϕ be as defined in Theorem 22. Then*

$$\|\mathcal{P}(\phi)(\lambda)\|_{HS} = 1$$

for a.e. $\lambda \in \mathbf{E} \cap \mathbf{C}$, and ϕ is an admissible vector for the representation $(L, \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$..

Proof. For any given $\lambda \in \mathbf{E} \cap \mathbf{C}$, there exists a unique indexing element j such that $\lambda \in \mathbf{E} \cap \mathbf{C} \cap A_j$ (see Lemma 16) and

$$\begin{aligned} \|\mathcal{P}(\phi)(\lambda)\|_{\mathcal{HS}}^2 &= \left\| (\mathbf{u}_\lambda^j \otimes \mathbf{e}_\lambda) |\det B(\lambda)|^{-1/2} \right\|_{\mathcal{HS}}^2 \\ &= |\det B(\lambda)|^{-1} \|\mathbf{u}_\lambda^j\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

From (5),

$$\|\mathcal{P}(\phi)(\lambda)\|_{\mathcal{HS}}^2 = |\det B(\lambda)|^{-1} \left\| |\det B(\lambda)|^{1/2} \chi_{E(\lambda)+k_j} \right\|_{L^2(\mathbb{R}^d)}^2 = 1.$$

Thus, ϕ is an admissible vector (see [6]) for the representation $(L, \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$. \square

Recall that

$$W_\phi : \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}) \rightarrow L^2(N), \quad W_\phi \psi(x) = \langle \psi, \pi(x) \phi \rangle.$$

Theorem 25. *Let ϕ be as defined in Theorem 22. The Hilbert subspace $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ is a sampling space and has the interpolation property. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is a sampling space with sinc-type function $W_\phi(\phi)$, and ϕ is constructed in Proposition 23.*

Proof. The proof follows directly from Proposition 3, Proposition 4 and Theorem 22. \square

Example 26. *Let N be a nilpotent Lie group with Lie algebra \mathfrak{n} spanned by the following fixed basis*

$$\{Z_1, Z_2, Y_1, Y_2, X_1, X_2\}$$

such that we have the following non-trivial Lie brackets

$$[X_1, Y_1] = [X_2, Y_2] = \frac{1}{5}Z_1, \quad [X_2, Y_1] = -\frac{1}{5}Z_2$$

and $[X_1, Y_2] = \frac{1}{5}Z_2$. Put

$$\Gamma = \prod_{k=1}^2 \exp(\mathbb{Z}Z_k) \prod_{k=1}^2 \exp(\mathbb{Z}Y_k) \prod_{k=1}^2 \exp(\mathbb{Z}X_k).$$

The Plancherel measure is supported on the manifold.

$$\Sigma = \left\{ \begin{array}{l} \lambda \in \mathfrak{n}^* : \lambda(Z_1)^2 + \lambda(Z_2)^2 \neq 0 \\ \lambda(Y_j) = 0, \lambda(X_j) = 0 \text{ for } 1 \leq j \leq 2 \end{array} \right\}.$$

Now put $\mathbf{E} = \{(\lambda_1, \lambda_2, 0, \dots, 0) \in \Sigma : \lambda_1^2 + \lambda_2^2 \leq 25\}$ and

$$\mathbf{E}^\circ = \left\{ \begin{array}{l} (\lambda_1, \lambda_2, 0, \dots, 0) \in \mathbf{E} : \\ -\left(\frac{3}{2}\right)^{1/4} 5^{1/2} \leq \lambda_1, \lambda_2 \leq \left(\frac{3}{2}\right)^{1/4} 5^{1/2} \end{array} \right\}.$$

It is not too hard to see that for all $\lambda \in \mathbf{E}^\circ$, $\|B(\lambda)^{tr}\| < 1$. Thus by Theorem 22, there exists a function in $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$ whose set of translates by elements of Γ forms an orthonormal basis. The set \mathbf{E}° is by no means unique. In fact, we could also choose

$$\tilde{\mathbf{E}} = \left\{ (\lambda_1, \lambda_2, 0, \dots, 0) \in \mathbf{E} : x^2 + y^2 \leq 5 \left(\frac{2}{\pi}\right)^{1/2} \right\}.$$

The Hilbert subspaces

$$\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \mathcal{P}^{-1} \left(\int_{\mathbf{E}^\circ}^{\oplus} (L^2(\mathbb{R}^2) \otimes \mathbf{e}_\lambda) \frac{(\lambda_1^2 + \lambda_2^2) d\lambda_1 d\lambda_2}{25} \right),$$

and

$$\mathbf{H}(\mathbf{e}, \tilde{\mathbf{E}}) = \mathcal{P}^{-1} \left(\int_{\tilde{\mathbf{E}}}^{\oplus} (L^2(\mathbb{R}^2) \otimes \mathbf{e}_\lambda) \frac{(\lambda_1^2 + \lambda_2^2) d\lambda_1 d\lambda_2}{25} \right)$$

are sampling spaces with interpolation properties with respect to Γ .

Example 27. Let N be a nilpotent Lie group with Lie algebra \mathfrak{n} spanned by the following fixed basis:

$$\{Z_1, Z_2, Y_1, Y_2, Y_3, X_1, X_2, X_3\}$$

with non trivial Lie brackets.

$$\begin{aligned} [X_1, Y_1] &= [X_3, Y_1] = \frac{1}{10} Z_1 \\ [X_3, Y_2] &= [X_3, Y_3] = \frac{1}{10} Z_1, \\ [X_2, Y_1] &= [X_2, Y_2] = \frac{1}{10} Z_2. \end{aligned}$$

Let $\pi : N \rightarrow GL(9, \mathbb{R})$ be a monomorphism such that for

$$p = \prod_{k=1}^2 \exp(z_k Z_k) \prod_{k=1}^3 \exp(y_k Y_k) \prod_{k=1}^3 \exp(x_k X_k)$$

we define

$$\pi(p) = \begin{bmatrix} 1 & 0 & \frac{x_1+x_3}{10} & \frac{x_3}{10} & \frac{x_3}{10} & -\frac{y_1}{10} & 0 & -\frac{y_1+y_2}{10} & z_1 \\ 0 & 1 & \frac{x_2}{10} & \frac{x_2}{10} & 0 & 0 & -\frac{y_1+y_2}{10} & 0 & z_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

π is a faithful matrix representation of N . Let

$$\Gamma = \prod_{k=1}^2 \exp(\mathbb{Z}Z_k) \prod_{k=1}^3 \exp(\mathbb{Z}Y_k) \prod_{k=1}^3 \exp(\mathbb{Z}X_k).$$

Moreover, the Plancherel measure is supported on the manifold

$$\Sigma = \left\{ \begin{array}{l} \lambda \in \mathfrak{n}^* : \lambda(Z_1)^2 \lambda(Z_2) \neq 0, \\ \lambda(Y_j) = 0, \lambda(X_j) = 0 \text{ for } 1 \leq j \leq 3 \end{array} \right\}.$$

Let $\mathbf{a} = 2^{2/5}3^{1/5}5^{3/5}$, and defining

$$\mathbf{E}^\circ = \left\{ \begin{array}{l} \lambda \in \Sigma : \lambda(Z_1)^2 |\lambda(Z_2)| \leq 1000 \\ -\mathbf{a} \leq \lambda(Z_1), \lambda(Z_2) \leq \mathbf{a} \end{array} \right\},$$

the Hilbert subspace

$$\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \mathcal{P}^{-1} \left(\int_{\mathbf{E}^\circ}^{\oplus} (L^2(\mathbb{R}^3) \otimes \mathbf{e}_\lambda) \frac{\lambda_1^2 |\lambda_2|}{1000} d\lambda_1 d\lambda_2 \right)$$

is a sampling space with interpolation property with respect to Γ .

Example 28. Let N be a nilpotent Lie group with Lie algebra \mathfrak{n} spanned by the following fixed basis:

$$\{Z_1, Z_2, Y_1, Y_2, Y_3, Y_4, X_1, X_2, X_3, X_4\}$$

with non trivial Lie brackets.

$$\begin{aligned} [X_1, Y_1] &= \frac{1}{70} Z_1, [X_1, Y_2] = \frac{1}{70} Z_2, [X_2, Y_1] = -\frac{1}{70} Z_2 \\ [X_2, Y_2] &= \frac{1}{70} Z_1, [X_3, Y_3] = \frac{1}{70} Z_2, [X_3, Y_4] = \frac{1}{70} Z_1 \\ [X_4, Y_3] &= -\frac{1}{70} Z_1, [X_4, Y_4] = \frac{1}{70} Z_2, [X_5, Y_5] = \frac{1}{70} Z_1 \\ [X_5, Y_6] &= \frac{1}{70} Z_2, [X_6, Y_5] = \frac{1}{70} Z_2, [X_6, Y_6] = \frac{1}{70} Z_1. \end{aligned}$$

Let

$$\Gamma = \prod_{k=1}^2 \exp(\mathbb{Z} Z_k) \prod_{k=1}^4 \exp(\mathbb{Z} Y_k) \prod_{k=1}^4 \exp(\mathbb{Z} X_k).$$

Defining

$$\mathbf{r}(\lambda(Z_1), \lambda(Z_2)) = \frac{(\lambda(Z_1)^2 - \lambda(Z_2)^2)(\lambda(Z_1)^2 + \lambda(Z_2)^2)^2}{15625},$$

and $\mathbf{a} = \left(\frac{3}{13}\right)^{1/8} 2^{3/8} 35^{3/8}$, we obtain the support of the Plancherel transform as follows:

$$\Sigma = \left\{ \begin{array}{l} \lambda \in \mathfrak{n}^* : |\mathbf{r}(\lambda(Z_1), \lambda(Z_2))| \neq 0, \\ \lambda(Y_j) = 0, \lambda(X_j) = 0 \text{ for } 1 \leq j \leq 4 \end{array} \right\}.$$

Defining

$$\mathbf{E}^\circ = \left\{ \begin{array}{l} \lambda \in \Sigma : |\mathbf{r}(\lambda(Z_1), \lambda(Z_2))| \leq 1 \\ -\mathbf{a} \leq \lambda(Z_1), \lambda(Z_2) \leq \mathbf{a} \end{array} \right\},$$

the Hilbert subspace

$$\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \mathcal{P}^{-1} \left(\int_{\mathbf{E}^\circ}^{\oplus} (L^2(\mathbb{R}^3) \otimes \mathbf{e}_\lambda) |\mathbf{r}(\lambda_1, \lambda_2)| d\lambda_1 d\lambda_2 \right)$$

is a sampling space with interpolation property with respect to Γ .

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